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On the rapid rotation of a massive sphere in a monatomic gas

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We consider the rapid rotation, in the sense of large Reynolds number ϵ^{-1}, of a gravitating solid sphere in a monatomic gas. The flow is characterized by a thin boundary layer on the sphere and a thin, swirling, buoyant, radial jet in the equatorial plane. When the Prandtl number σ is of order unity, the boundary layer and jet are to first order uncoupled from the outer flow. For sufficiently small Prandtl number (which may be interpreted as approximating an optically thick radiating gas), the outer flow *is* boundary-layer driven. The parameter boundary between these two dissimilar steady states is $\sigma = O(\epsilon^{\frac{1}{3}})$.

1. Introduction

We consider here the rapid rotation of a massive and hence gravitating solid sphere in a monatomic gas otherwise at rest. Rapid rotation here means large Reynolds number $\Omega a^2/\nu_0 = \epsilon^{-1}$, where Ω is the angular velocity of the sphere, a its radius and ν_0 is the kinematic viscosity at the surface. The structure of the flow is shown schematically in figure 1. Adjacent to the spherical surface, denoted henceforth by $\mathscr{S}$, is a thin viscous boundary layer of width $(\nu_0/\Omega)^{\frac{1}{2}}$; the fluid entrained by this boundary layer erupts from $\mathscr{S}$ in a thin swirling radial jet in the equatorial plane, denoted by $\mathscr{E}$, which has initial width $(\nu_0/\Omega)^{\frac{1}{2}}$ as well.

These basic features of the flow are those of a rotating sphere in an incompressible fluid also. Howarth (1951) first wrote down the appropriate boundary-layer equations and gave a power-series solution. Squire (1955) discussed the structure of such a radial jet in another context, though swirl was not included. Stewartson (1958) noted the proper way in which to join the boundary-layer solution to the jet at the intersection of $\mathscr{S}$ and $\mathscr{E}$. There has been some conjecture that the boundary layer developing from the pole separates either before or just at $\theta = \frac{1}{2}\pi$ (Nigam 1954; Singh 1970), but Stewartson (1958) and Foster (1972) have presented arguments that the boundary layer is attached on the interval $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The numerical solutions of Fox (1964), Banks (1965) and Manohar (1967) all support this latter view; in fact, if the boundary layer developing from one pole is somehow removed, Fox's results predict a point of zero skin friction at about 100°.

The jet on $\mathscr{E}$ presents certain difficulties which may be understood as follows. The fluid is entrained by the boundary layer at a temperature which is the fluid

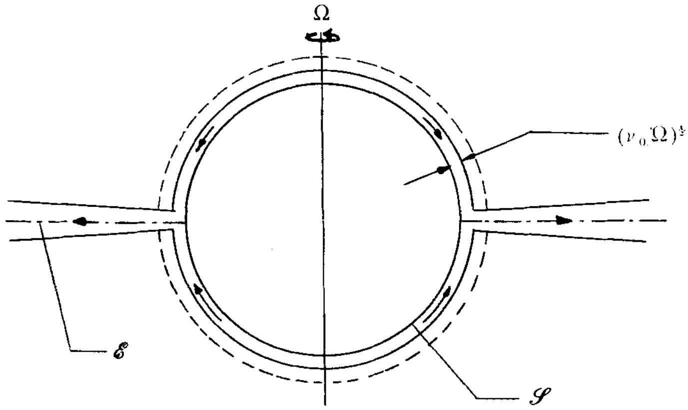

FIGURE 1. Schematic representation of the flow induced by the rapid rotation of a massive sphere in a gas.

temperature at the boundary-layer edge. Once in the boundary layer, the fluid is transported towards $\mathscr{E}$ and during that process is heated by viscous dissipation and either gains or loses heat to the solid surface (unless the surface is insulated). When the boundary-layer material finally arrives at $\mathscr{E}$, it is either colder or hotter than the fluid just outside the boundary layer. If it is hotter, buoyancy forces accelerate the jet; if it is colder, the retarding effects of this cold fluid moving through warm surroundings might possibly terminate the jet. Should the jet terminate at a finite distance from the sphere, the general model of the flow presented herein would be breakdown. In § 6, the condition for jet penetration to infinity is given.

Except in the regions of large gradients near $\mathscr{S}$ and $\mathscr{E}$ discussed above, the flow is inviscid, but the structure is quite different when the Prandtl number is of order unity from the structure given in Foster (1970, hereafter referred to as I), for the small-Prandtl-number flow. In fact, the critical parameter is found, in § 7, to be $\sigma/\epsilon^{\frac16}$. If $\sigma = o(\epsilon^{\frac16})$, the outer flow is boundary-layer driven and the velocities are $O(\Omega a\epsilon^{\frac12})$; the streamlines, from I, are shown in figure 2(*a*). If $\sigma > O(\epsilon^{\frac16})$, § 5 shows the streamlines to have the general form given in figure 2(*b*). Velocities in that range are much larger, $O(\Omega a(\epsilon/\sigma)^{\frac13})$, and are associated with the free convection of the fluid. The somewhat surprising thing here is that the matching (see § 4) indicates that this convective motion is uncoupled to the jet and the boundary layer, to first order !†

2. Formulation and outer expansion

The equations of motion for the problem set in § 1 are given below in non-dimensional form for convenience. All thermodynamic quantities are based on their values on the surface $\mathscr{S}$; this is the simplest procedure though the pressure

† It should be noted that, if the sphere is non-rotating, the free convection velocities along the surface must be brought to zero through the action of viscosity. In that case, the boundary layer has width $(a^2\nu_0\sigma^{\frac12}/c_0)^{\frac13}$, $\propto \nu_0^{\frac13}$, where c_0 is the isothermal sound speed. The rotation, then, thins the layer from a width $\propto \nu_0^{\frac13}$ to a width $\propto \nu_0^{\frac12}$.

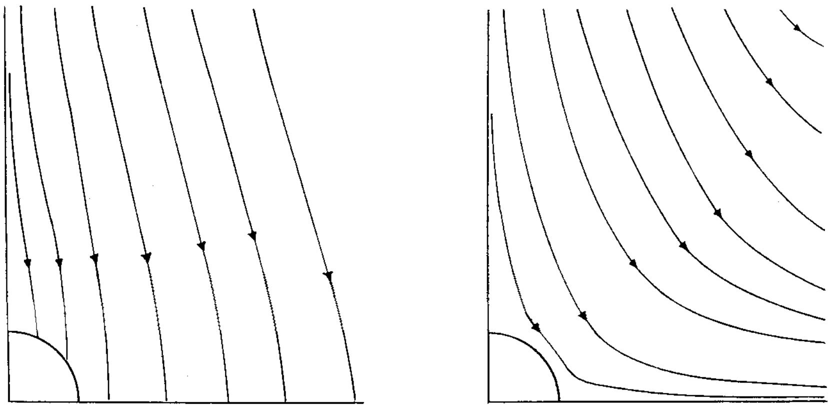

FIGURE 2. Meridional plane outer flow. (a) $\sigma = o(\epsilon^{\frac{1}{6}})$. (b) $O(\epsilon^{\frac{1}{6}}) < \sigma \leqslant O(1)$.

at infinity is the real pressure scale. Basing velocities on the equatorial speed Ωa, we have then (Lagerstrom 1964)

$$\nabla.(\rho\mathbf{u}) = 0, \tag{2.1a}$$

$$B^2\rho(\mathbf{u}.\nabla)\mathbf{u}+\nabla p = \rho\nabla(\alpha/r)+\epsilon B^2\nabla.\boldsymbol{\tau}, \tag{2.1b}$$

$$\rho\mathbf{u}.\nabla T-\tfrac{2}{5}\mathbf{u}.\nabla p = (\epsilon/\sigma)\nabla.(T^{\omega}\nabla T)+\tfrac{2}{5}\epsilon B^2\Phi_v, \tag{2.1c}$$

$$p = \rho T, \tag{2.1d}$$

$$\boldsymbol{\tau} = T^{\omega}(\mathbf{e}-\tfrac{2}{3}\mathbf{I}\nabla.\mathbf{u}), \tag{2.1e}$$

where $\mathbf{e}$ is the deformation tensor, $\frac{1}{2} \leqslant \omega \leqslant 1$ for a monatomic gas (Chapman & Cowling 1952, p. 220) and Φ_v is the viscous dissipation. We use spherical polar co-ordinates (r,θ,ϕ) throughout; the velocity components are (u,v,w) respectively. The four parameters that occur are

$$\epsilon = \nu_0/\Omega a^2, \quad \sigma = \nu_0/\kappa_0, \quad B = \Omega a/c_0, \quad \alpha = ag_0/c_0^2,$$

where ν_0, κ_0, c_0 and g_0 are respectively the kinematic viscosity, thermometric conductivity, isothermal sound speed and gravitational acceleration on the surface $\mathscr{S}$.

Equations (2.1) are to be solved, then, subject to the boundary conditions

$$\left.\begin{aligned} \mathbf{u} = \mathbf{k}\times\mathbf{r}, \quad p = 1 \\ T = 1 \quad \text{or} \quad \mathbf{r}.\nabla T = 0 \end{aligned}\right\} \quad \text{on} \quad \mathscr{S}, \tag{2.2a}$$

where $\mathbf{k} = \boldsymbol{\Omega}/\Omega$, and

$$\mathbf{u}\rightarrow 0, \quad T\rightarrow T_{\infty}, \quad p \sim \text{static atmosphere as } r\rightarrow\infty. \tag{2.2b}$$

In this paper, we seek the uniformly valid asymptotic solution for $\epsilon \downarrow 0$ with B and α of order unity; the order of magnitude of σ presents something more of a problem. In I, solutions for $\epsilon \downarrow 0$ were constructed for $\sigma = o(\epsilon^{\frac{1}{6}})$.† Here, we cover the previously unsolved range, $O(1) \geqslant \sigma > O(\epsilon^{\frac{1}{6}})$.

† In I, it is pointed out that $\sigma = o(1)$ may be interpreted physically as a Prandtl number associated with radiation heat transfer in an optically thick gas.

Since motion occurs about a spinning sphere only because the fluid has viscosity, it appears that $|\mathbf{u}| = o(1)$ in ϵ away from any thin viscous layers. So, to leading order, the outer solution is hydrostatic, i.e.

$$\nabla p_1 = \rho_1 \nabla(\alpha/r). \tag{2.3}$$

The additional information needed to compute $p_1 = p_1(r)$ comes from the energy transport characteristics in the fluid, i.e. the limit of (2.1*c*) as $\epsilon \downarrow 0$. That is not trivial to determine since the order of magnitude of $|\mathbf{u}|$ must be known. Provided that the order of $|\mathbf{u}|$ is greater than ϵ/σ, which must be checked *a posteriori*, (2.1*c*) yields

$$\rho_1 \mathbf{u}_1 . \nabla T_1 - \tfrac{2}{5}\mathbf{u}_1 . \nabla p_1 = 0. \tag{2.4}$$

So long as the outer flow streamlines originate at infinity, which is proved in §5, solution of (2.1*d*), (2.3) and (2.4) gives

$$T_1 = T_\infty(1+\beta/r), \quad p_1 = p_\infty(1+\beta/r)^{\frac{5}{2}}, \tag{2.5}$$

where $\beta \equiv 2\alpha/5T_\infty$. This solution corresponds to what is generally referred to as an 'adiabatic atmosphere'.

The velocity scale of the outer flow presents some difficulty. As will be shown in §3, the boundary-layer entrainment is of order $\epsilon^{\frac{1}{2}}$; the assumption that the outer flow has this scale, an assumption used in I throughout, leads to a unique uniformly valid solution only if $\sigma = o(\epsilon^{\frac{1}{2}})$. So long as α is $O(1)$, there are strong convective forces which drive a motion with velocity magnitude $(\epsilon/\sigma)^{\frac{1}{3}} \gg \epsilon^{\frac{1}{2}}$. Hence, we now write the outer expansion as

$$\left.\begin{aligned} \mathbf{u} &= (\epsilon/\sigma B^2)^{\frac{1}{3}}\mathbf{u}_1 + \epsilon^{\frac{1}{2}}\mathbf{u}_2 + \ldots, \\ p &= p_1 + (\epsilon B/\sigma)^{\frac{2}{3}} p_2 + \ldots. \end{aligned}\right\} \tag{2.6}$$

Substitution into (2.1) does indeed produce (2.3) and (2.4) to first order, and to next order,

$$\nabla . (\rho_1 \mathbf{u}_1) = 0, \tag{2.7a}$$

$$\rho_1(\mathbf{u}_1 . \nabla)\mathbf{u}_1 + \nabla p_2 = \rho_2 \nabla(\alpha/r), \tag{2.7b}$$

$$\rho_1 \mathbf{u}_1 . \nabla H = (\omega+1)^{-1}\nabla^2 T_1^{\omega+1}, \tag{2.7c}$$

$$p_2 = \rho_2 T_1 + \rho_1 T_2, \quad H = T_2 + \tfrac{1}{5}|\mathbf{u}_1|^2. \tag{2.7d, e}$$

We note that the azimuthal component of (2.7*b*) is

$$\rho_1 \mathbf{u}_1 . \nabla(r w_1 \sin\theta) = 0,$$

which has the solution $w_1 = 0$. We note further that the velocity scale

$$\Omega a(\epsilon/\sigma B^2)^{\frac{1}{3}} = (\nu_0 c_0^2/a\sigma)^{\frac{1}{3}},$$

which is Ω independent as it must be since, as we shall show in §4, this motion is uncoupled to the rotation of the sphere and will occur whether or not the sphere rotates.

We postpone further discussion of (2.7) until §5, since boundary conditions must be carefully deduced by matching with the boundary layer on $\mathscr{S}$ described in the next section.

It should be noted that the series (2.6) is indeed asymptotic to the solution as $\epsilon \downarrow 0$ only if α is of sufficient size, i.e. $\mathbf{u}_1 = \mathbf{u}_1(\mathbf{r};\alpha)$, and, from (5.3), we note that $|\mathbf{u}_1| \propto \alpha$ as $\alpha \downarrow 0$. Therefore, if (2.6) is to be an asymptotic series, we require

$$\alpha \gg \epsilon^{\frac{1}{6}}\sigma^{\frac{1}{3}}B^{-\frac{2}{3}}.$$

However, if α is too large, pressure gradients normal to the sphere will invalidate the analysis of § 3. Hence, we also require that $\alpha \ll \epsilon^{-\frac{1}{2}}$.

3. The boundary layer; $\sigma = O(1)$

Since the velocity in the outer flow is $o(1)$ and the boundary condition (2.2a) involves velocities $O(1)$, there must be a viscous boundary layer on $\mathscr{S}$. If $\sigma = O(1)$, the layer is compressible and has width $\epsilon^{\frac{1}{2}}$; $y = (r-1)/\epsilon^{\frac{1}{2}}$ is the boundary-layer co-ordinate. Under a Howarth–Dorodnitsyn transformation (see Stewartson 1964, p. 29),

$$\left.\begin{aligned} y &= \int_0^\eta \hat{T}(\lambda,\theta)\,d\lambda, \\ \hat{u} &= \epsilon^{\frac{1}{2}}\Big(\hat{T}\hat{u}^* - \hat{v}\hat{T}\ \frac{\partial}{\partial\theta}\int_0^y \hat{\rho}(\tilde{y},\theta)\,d\tilde{y}\Big), \end{aligned}\right\} \tag{3.1}$$

the boundary-layer equations become

$$\sin\theta\,\hat{u}^*_\eta + (\hat{v}\sin\theta)_\theta = 0, \tag{3.2a}$$

$$L\hat{v} - \hat{w}^2\cot\theta = (\hat{T}^{\omega-1}\hat{v}_\eta)_\eta, \tag{3.2b}$$

$$L(\hat{w}\sin\theta) = (\hat{T}^{\omega-1}\hat{w}_\eta\sin\theta)_\eta, \tag{3.2c}$$

$$L\hat{T} = \sigma^{-1}(\hat{T}^{\omega-1}\hat{T}_\eta)_\eta + \tfrac{2}{5}B^2\hat{T}^{\omega-1}(\hat{v}^2_\eta + \hat{w}^2_\eta), \tag{3.2d}$$

$$L \equiv \hat{u}^*\,\partial/\partial\eta + \hat{v}\partial/\partial\theta, \tag{3.2e}$$

where the caret refers to a boundary-layer variable. The boundary conditions (2.2a) are here

$$\hat{u}^* = \hat{v} = 0, \quad \hat{w} = \sin\theta; \quad \hat{T} = 1 \quad \text{or} \quad \hat{T}_\eta = 0 \quad \text{on} \quad \eta = 0, \tag{3.3}$$

and matching with (2.5) and (2.6) obviously requires of the *leading-order* solutions of (3.2) that

$$\hat{v}, \hat{w} \to 0, \quad \hat{T} \sim T_\infty(1+\beta) \quad \text{as} \quad \eta \to \infty. \tag{3.4}$$

An immediate consequence of (3.4) is

$$y \sim T_\infty(1+\beta)\,\eta \quad \text{as} \quad \eta \to \infty,$$

so that the layer thickness is not $\epsilon^{\frac{1}{2}}$ but rather $\epsilon^{\frac{1}{2}}(1+\beta)T_\infty$. We note that in the special case $\omega = 1$, a 'hard sphere gas' (Chapman & Cowling 1952, p. 30), equations (3.2) are decoupled and the velocity components are those for the incompressible boundary layer in the η, θ plane. There are quite accurate numerical solutions in the literature (Banks 1965; Manohar 1967) for the incompressible problem, and these may be used to integrate the energy equation numerically. Once $\hat{T}$ has been found, the transformation may be inverted to find the solutions as functions of (y, θ).

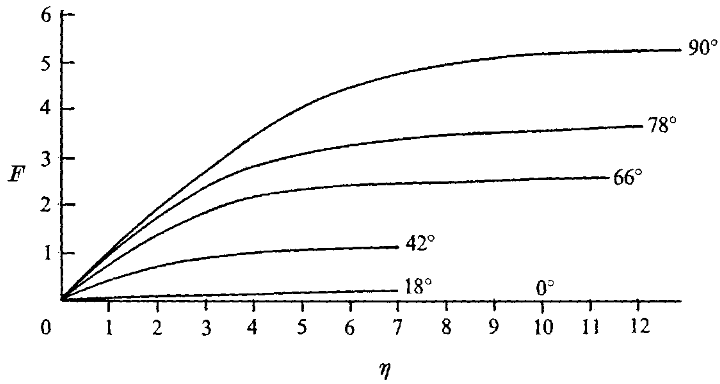

FIGURE 3. Compressibility co-ordinate correction for an insulated sphere at various values of the polar angle θ; $\omega = 1$.

It may easily be shown by writing down a total temperature equation from (3.2) that, provided $\sigma = 1$,

$$\hat{T} = T_\infty(1+\beta)+\tfrac{1}{5}B^2(\hat{v}^2+\hat{w}^2)+\tfrac{2}{5}B^2\hat{w}\sin\theta$$

is the temperature solution for the case when the sphere is thermally insulated ($\hat{T}_\eta = 0$ on $\eta = 0$); note that the surface temperature has a $\sin^2\theta$ distribution. Using the results of Manohar (1967), $\hat{T}$ was computed, then the integral in (3.1) was calculated numerically and hence, provided $\omega = 1$ as well,

$$y = T_\infty(1+B)\,\eta+\tfrac{1}{5}B^2F(\eta,\theta),$$

where F is shown in figure 3 as a function of η for various values of θ.

The boundary layers developing from each pole of the sphere collide at $\theta = \frac{1}{2}\pi$ and form a flat swirling radial jet on $\mathscr{E}$ (Stewartson 1958; Foster 1972). All along their length the boundary layers entrain fluid, viz.

$$u \sim -\epsilon^{\frac{1}{2}}T_\infty(1+\beta)\,E_\omega(\theta) \quad \text{as} \quad y\to\infty. \tag{3.5}$$

Note that $E_1(\theta) \equiv E(\theta)$, which is the entrainment rate for the incompressible boundary layer (tabulated in Manohar 1967); further, for $\omega \neq 1$, $E_\omega(\theta)$ will also depend explicitly on B^2.

4. The matching; $\sigma = O(1)$

Before giving details of the structure of the free convection, it is necessary to study the character of the matching of the inner (boundary-layer) expansion and the outer expansion for $\sigma = O(1)$.

Recall from (2.6) that the outer expansion proceeds

$$\mathbf{u} = \gamma_1\mathbf{u}_1+\gamma_2\mathbf{u}_2+\ldots, \tag{4.1}$$

where we have written, for convenience, γ_1 for $(\epsilon/\sigma B^2)^{\frac{1}{2}}$ and γ_2 for $\epsilon^{\frac{1}{2}}$. The boundary-layer expansion is

$$\left.\begin{aligned} u &= \gamma_2\hat{u}_1+\ldots,\\ (v,w) &= (\hat{v}_1,\hat{w}_1)+\mu(\hat{v}_2,\hat{w}_2)+\ldots,\end{aligned}\right\} \tag{4.2}$$

where a caret again denotes a function of (y, θ) and $(\hat{u}_1, \hat{v}_1, \hat{w}_1)$ is the *leading-order* boundary-layer solution discussed in § 3. $\mu = o(1)$, but is otherwise as yet undetermined.

Because the matching must be done with some care, we use the 'intermediate limit' (Cole 1968, p. 17); let $y_\delta \equiv (r-1)/\delta$, where δ is such that $O(\epsilon^{\frac{1}{2}}) < \delta = o(1)$. If we denote by 'lim' the limit δ, $(\epsilon^{\frac{1}{2}}/\delta) \downarrow 0$ with y_δ fixed, then

$$\lim \{(\gamma_2 \hat{u}_1 - \gamma_1 u_1)/\gamma_1\} = -u_1(1, \theta) = 0,$$

so that the boundary condition for (2.7) is $u_1 = 0$ on $\mathscr{S}$. Clearly, (3.4) already involves some anticipation of the matching principles since

$$\lim (\hat{v}_1 - \gamma_1 v_1) = \lim (\hat{w}_1 - \gamma_1 w_1) = 0.$$

Therefore, the expansions match to first order if

$$u_1 = 0 \quad \text{on} \quad \mathscr{S}. \tag{4.3}$$

We note also that the radial jet on $\mathscr{E}$ entrains fluid at the same rate as the boundary layer, so that a similar matching procedure applied on $\mathscr{E}$ will yield

$$v_1 = 0 \quad \text{on} \quad \mathscr{E}. \tag{4.4}$$

A second application of the limit to the radial velocity u gives

$$\lim [(\gamma_1 u_1 + \gamma_2 u_2 + \ldots - \gamma_2 \hat{u}_1)/\gamma_2] = u_2(1, \theta) - \hat{u}_1(\infty, \theta) - \frac{\delta \gamma_1}{\gamma_2} \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} (v_1(1, \theta) \sin\theta), \tag{4.5}$$

and so long as $\delta = o(\epsilon^{\frac{1}{6}})$, which still allows overlap,

$$u_2 = \hat{u}_1(\infty, \theta) = -E_\omega T_\infty (1 + \beta) \quad \text{on} \quad \mathscr{S}. \tag{4.6}$$

A second limit applied to v and w yields matching conditions for higher-order boundary-layer solutions; if $\mu = \gamma_1$, then

$$\hat{v}_2(\infty, \theta) = v_1(1, \theta), \quad \hat{w}_2(\infty, \theta) = w_1(1, \theta). \tag{4.7}$$

There will also be a boundary condition on the $\mathbf{u}_2$ flow on $\mathscr{E}$, much like (4.6).

5. The structure of the free convection

From § 4, the free convection equations (2.7) are to be solved with the boundary conditions

$$\mathbf{r} \cdot \mathbf{u}_1 = 0 \quad \text{on} \quad \mathscr{S}, \quad \mathbf{k} \cdot \mathbf{u}_1 = 0 \quad \text{on} \quad \mathscr{E}, \quad |\mathbf{u}_1| \to 0 \quad \text{as} \quad r \to \infty. \tag{5.1}$$

Solutions have not as yet been obtained to this boundary-value problem. We give below some properties of the solution.

The streamline pattern

We shall prove that there are no closed streamlines in this flow and that, hence, the streamline pattern must be generally of the form shown in figure 2 (*b*); the proof is by contradiction.

Since $\mathbf{u}_1 = \mathbf{u}_1(r,\theta)$, it is sufficient to consider a streamline in the r, θ plane, say C_ψ, that encloses a finite region R. Equation (2.7*c*) is

$$\nabla.(\rho_1 \mathbf{u}_1 H) = [2\alpha/5r^2]^2\,\omega\, T_1^{\omega-1}$$

and integration over R with use of Gauss's theorem (Jeffreys & Jeffreys 1956, p. 211) gives

$$\int_{C\psi} \rho_1 H(\mathbf{u}_1.\mathbf{n})\,dl = \int_R [2\alpha/5r^2]^2\,\omega T_1^{\omega-1}\,d\tau.$$

The integrand on the left is zero by definition of C_ψ even if certain portions of C_ψ consist of segments of $\mathscr{S}$ and $\mathscr{E}$, since, by (5.1), $\mathscr{S}$ and $\mathscr{E}$ are also streamlines of the $\mathbf{u}_1$ flow; therefore, provided that $\omega\alpha^2 \not\equiv 0$, we have

$$\int_R \frac{T_1^{\omega-1}}{r^4}\,d\tau = 0, \tag{5.2}$$

which is impossible since the integral is positive definite. Thus there is a contradition and the original assumption is violated. There can be no closed streamlines in the first-order outer flow.

Asymptotic solution as $r \to \infty$

Note that, since $T_1 = T_\infty(1+O(\alpha/r))$, $T_1 \simeq T_\infty$ for $\alpha \downarrow 0$ and all $r \geqslant 1$ or for α arbitrary and r sufficiently large. In that eventuality, we write

$$\left.\begin{aligned} u_1 = -\alpha c\psi_\theta^*/r^2\sin\theta,\quad v_1 = \alpha c\psi_r^*/r\sin\theta,\\ T_2 = \alpha c^2 T_\infty T^*,\quad c \equiv (4\omega T_\infty^{\omega-1}/25p_\infty)^{\frac{1}{2}},\end{aligned}\right\} \tag{5.3}$$

and then (2.7) reduce to

$$\left.\begin{aligned} \frac{\partial(\psi^*,\omega^*)}{\partial(r,\theta)} = \frac{1}{r}\frac{\partial T^*}{\partial\theta},\quad \frac{\partial(\psi^*,T^*)}{\partial(r,\theta)} = \frac{\sin\theta}{r^2},\\ \omega^* r\sin^2\theta = -\psi_{rr}^* - \frac{\sin\theta}{r^2}\left(\frac{\psi_\theta^*}{\sin\theta}\right)_\theta.\end{aligned}\right\} \tag{5.4}$$

In particular, these equations admit the asymptotic solutions

$$\psi^* \sim g(\theta)\,r^{\frac{2}{3}},\quad T^* \sim f(\theta)\,r^{-\frac{5}{3}}\quad \text{as}\quad r\to\infty.$$

The functions f and g satisfy the ordinary differential equations

$$2gf' + 5fg' = 3\sin\theta,\quad 3f' + 2gQ' + 7Qg' = 0, \tag{5.5}$$

where $Q(\theta)$ is the angular variation of the vorticity ω^* and is given by

$$Q(\theta) = [g'' - g'\cot\theta - \tfrac{2}{9}g]/\sin^2\theta.$$

The boundary conditions are

$$g(0) = g'(0) = g(\tfrac{1}{2}\pi) = 0.$$

The consequence of the results here is that all streamlines of the $\mathbf{u}_1$ flow must originate at infinity, where $T_1 = T_\infty$, thus proving the validity of (2.5). (If closed streamlines could exist, (2.5) could not be correct for all $r < \infty$.)

6. The equatorial jet

The fluid entrained by the viscous boundary layer on $\mathscr{S}$, discussed in § 3, erupts at $\theta = \frac{1}{2}\pi$ into a thin swirling radial jet concentrated near $\mathscr{E}$. For $\sigma = O(1)$, buoyancy forces are significant to the dynamics of the jet, and a self-similar solution is in general impossible. In fact, very large negative buoyancy will cause the jet to terminate at a finite distance $r = r_*$ from the sphere.

If the Howarth–Dorodnitsyn transformation is made the jet equations are

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)+\frac{1}{r}\frac{\partial v^\dagger}{\partial \phi} = 0, \tag{6.1a}$$

$$\mathscr{L}u-\frac{w^2}{r}+\frac{\alpha}{B^2 T_1 r^2}(T_1-T) = 0, \tag{6.1b}$$

$$\mathscr{L}(rw) = 0, \tag{6.1c}$$

$$\mathscr{L}(T_t-T_1)-\left(\frac{1}{\sigma}-1\right)\frac{C}{r^2(\omega+1)}\frac{\partial^2 T^{\omega+1}}{\partial \phi^2} = 0, \tag{6.1d}$$

where $v^\dagger$ is a transformed θ velocity, T_t is the total temperature $T+\frac{1}{5}B^2(u^2+w^2)$, $C \equiv (T_\infty/T_1)^{\frac{1}{2}}$ and

$$\mathscr{L} = u\frac{\partial}{\partial r}+\frac{v^\dagger}{r}\frac{\partial}{\partial \phi}-C\frac{\partial}{\partial \phi}T^{\omega-1}\frac{\partial}{\partial \phi}. \tag{6.1e}$$

The physical-plane boundary-layer variable $\bar{\theta}$ is

$$\bar{\theta} = \frac{\pi-\theta}{\epsilon^{\frac{1}{2}}} = \int_0^\phi \frac{T(r,s)\,ds}{T_1(r)}. \tag{6.2}$$

The third term in (6.1*b*) is the buoyancy term, which is critical to the question of jet penetration to infinity.

For the case $\sigma = O(1)$, it is proved in appendix A that a sufficient condition for such penetration of the jet through the 'adiabatic' atmosphere to infinity is

$$\int_{-\infty}^{\infty} u(T_t-T_1)\,d\phi|_{r=1} \geqslant 0. \tag{6.3}$$

Stewartson (1958) gave arguments for the way in which incompressible boundary layers on $\mathscr{S}$ developing from the poles collide at $\theta = \frac{1}{2}\pi$, and turn to form the radial jet on $\mathscr{E}$. This turning, as he noted, occurs in a thin annulus of radius 1 and cross-section $\epsilon^{\frac{1}{2}} \times \epsilon^{\frac{1}{2}}$. If the boundary layers are compressible, then the total temperature, entropy and angular momentum are invariant along streamlines through the turn. That being the case, (6.3) may be recast in a form that relates to the boundary-layer structure rather than to that of the jet, viz.

$$\int_0^\infty \hat{v}(\hat{T}-T_1(1)+\tfrac{1}{5}B^2(\hat{v}^2+\hat{w}^2))\,d\eta|_{\theta=\frac{1}{2}\pi} \geqslant 0. \tag{6.4}$$

The boundary-layer equations (3.2) may be integrated to yield an alternative version of (6.4):

$$\tfrac{2}{5}B^2\int_0^\infty \hat{v}\hat{w}\,d\eta|_{\theta=\frac{1}{2}\pi}-\frac{1}{\sigma}\int_0^{\frac{1}{2}\pi}\hat{T}^{\omega-1}\hat{T}_\eta|_{\eta=0}\sin\theta\,d\theta \geqslant 0. \tag{6.5}$$

The insulated sphere satisfies this requirement, provided that $\hat{v}$ and $\hat{w}$ are one-signed throughout.

If $\omega = 1$, the first integral in (6.5) may be done numerically from the data given in Manohar (1967), with the result that (6.5) takes the form

$$\int_0^{\frac{1}{2}\pi} \hat{T}_\eta|_{\eta=0} \sin\theta\, d\theta \leqslant 0{\cdot}11\sigma B^2, \quad \omega = 1. \tag{6.6}$$

From either (6.5) or (6.6), it is clear that, if the sphere is sufficiently cold, then $\hat{T}_\eta|_{\eta=0}$ will be sufficiently positive to violate this condition.

7. The solution structure for $\sigma = o(1)$

As previously mentioned, a unique solution is given in I for the parameter range $\sigma = o(\epsilon^{\frac{1}{6}})$ in which $\mathbf{u}_1 \equiv 0$ and $p_2 = T_2 = \rho_2 \equiv 0$, i.e. with no free convection. We explore here the reason for the disappearance of this convective motion as $\sigma \downarrow 0$.

When $\sigma = o(1)$, the equatorial jet cannot possibly terminate but exists intact to infinity; the self-similar structure of such a jet is given in § 4 of I. There are no buoyancy forces in the small-Prandtl-number jet because the relatively large heat conductivity allows the jet material to adjust to the local temperature of the fluid at the jet edge at every radial station.

The $\sigma = o(1)$ boundary-layer structure is essentially in two parts. The velocity boundary layer is incompressible, with width $\epsilon^{\frac{1}{2}}$, as in § 3, and it lies beneath, provided that $\sigma > O(\epsilon^{\frac{1}{2}})$, a thicker thermal layer of thickness $\epsilon^{\frac{1}{2}}/\sigma$. It is shown in appendix B that if $\sigma > O(\epsilon^{\frac{1}{6}})$ the thermal layer has a relatively simple structure described by

$$-E_\omega(\theta)\frac{\partial T}{\partial y_1} = \frac{\partial}{\partial y_1} T^\omega \frac{\partial T}{\partial y_1}, \tag{7.1}$$

where y_1 is the thermal-layer co-ordinate $(r-1)/(\epsilon^{\frac{1}{2}}/\sigma)$. The solution of this equation, as given in § 6 of I, is $T = 1$ if the sphere is insulated and

$$\left.\begin{aligned} \int_R^1 [T_\infty(1+\beta) - (T_\infty(1+\beta)-1)\,s]^\omega \frac{ds}{s} &= E_\omega(\theta)\, y_1, \\ R \equiv \frac{T - T_\infty(1+\beta)}{1 - T_\infty(1+\beta)}, & \end{aligned}\right\} \tag{7.2}$$

if $T = 1$ on $r = 1$.

For $O(\epsilon^{\frac{1}{2}}) < \sigma = o(\epsilon^{\frac{1}{6}})$, appendix B also shows that the matching of the outer expansion with the boundary layers requires that

$$\frac{d}{d\theta}\frac{1}{\sin\theta}\frac{d}{d\theta}[v_1(1,\theta)\sin\theta] = 0,$$

which can be true only if $u_1 = v_1 = p_2 = T_2 = \rho_2 = 0$, so the free convection disappears in that range of σ.

Thus, for $O(\epsilon^{\frac{1}{6}}) < \sigma = o(1)$ the convection exists and the flow is much like the $\sigma = O(1)$ flow described in §§ 2–6; for $O(\epsilon^{\frac{1}{2}}) < \sigma = o(\epsilon^{\frac{1}{6}})$, no convection can exist and the leading-order velocity scale is $\epsilon^{\frac{1}{2}}$. The details of the solution for this case may be found in I. (In I unique solutions are given for all $\sigma = o(\epsilon^{\frac{1}{3}})$, as mentioned above. The solutions were constructed there without consideration of any freely convecting flow; it was found, however, that solutions that tacitly fail to include free convection break down at a value of σ of $\epsilon^{\frac{1}{3}}$.)

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Appendix A

We consider the buoyant jet described by (6.1). In particular, (6.1*b*) may be integrated to give

$$\frac{dI}{dr}+\frac{\alpha}{5T_1 r^2}I = \frac{r+\frac{1}{2}\beta}{r+\beta}\frac{1}{r}\int_{-\infty}^{\infty} w^2\,d\phi+\frac{\alpha}{B^2 r^2 T_1}\int_{-\infty}^{\infty}(T_t-T_1)\,d\phi, \tag{A 1}$$

where

$$I \equiv \int_{-\infty}^{\infty} r^2 u^2\,d\phi,$$

the momentum flux. We note that the first term on the right side of the equation is the centrifugal force, which accelerates the jet towards infinity; the second term is the buoyancy force, which may be accelerative or decelerative, depending upon its sign. The quantity I is positive definite; should (A 1) indicate a simple zero of I, the jet would fail to exist at that point. A necessary condition for such a simple zero of I at, say, $r = r_*$ is, noting that I' must be negative there,

$$\int_{-\infty}^{\infty}(T_t-T_1)\,d\phi < -\frac{B^2 T_\infty}{\alpha}(r+\tfrac{1}{2}\beta)\int_{-\infty}^{\infty} w^2\,d\phi \quad \text{at} \quad r = r_*. \tag{A 2}$$

Now consider a function $f(r,\phi)$ that satisfies $\mathscr{L}f = 0$ in D: $r > 1$, $|\phi| < \infty$. If $f(1,\phi) \geqslant 0$, the maximum principle for such a parabolic equation (see Friedman 1964, chap. 2) states that $f \geqslant 0$ in D.

For $\sigma = 1$, T_t satisfies $\mathscr{L}T_t = 0$ in D, so clearly, if $T_t - T_1 \geqslant 0$ on $r = 1$, then $T_t - T_1 \geqslant 0$ in D; equation (A 2) then indicates that, if $T_t - T_1 \geqslant 0$ on $r = 1$, I has no zero, so that the jet exists intact to infinity. Conversely, if $T_t - T_1$ is negative on $r = 1$, equation (A 2) indicates the possibility that the jet will terminate.

By forming a kinetic energy equation from (6.1*b*) and (6.1*c*), one can easily show that the negative-definite dissipation term gives the bound

$$E(r) \leqslant E(1)+\frac{\alpha}{B^2 T_\infty(1+\beta)}(1-1/r)\int_{-\infty}^{\infty} u(T-T_1)\,d\phi|_{r=1}, \tag{A 3}$$

where

$$E = \int_{-\infty}^{\infty} r^2 u(\tfrac{1}{2}u^2+\tfrac{1}{2}w^2)\,d\phi,$$

from which we note that the jet will terminate or reverse if

$$\int_{-\infty}^{\infty} u(T_t-T_1)\,d\phi|_{r=1} \leqslant -\frac{B^2 T_\infty(1+\beta)}{\alpha}E(1). \tag{A 4}$$

In addition, we further shall show below that a self-similar solution of (6.1) is possible for $T_t - T_1 \equiv 0$. All of the above then demonstrates that

$$\int_{-\infty}^{\infty} u(T_t - T_1)\,d\phi|_{r=1} \geqslant 0$$

is a sufficient condition for existence of the jet to $r = \infty$.

Similarity solution

Equations (6.1) may be put in an especially useful form for a 'model fluid' (Stewartson 1964, p. 37), i.e. $\sigma = \omega = 1$,

$$\frac{\partial(\psi, \psi_\phi)}{\partial(\phi, r)} - \frac{1}{r}\psi_\phi^2 \frac{4r+3\beta}{2(r+\beta)} - rW^2\frac{2r+\beta}{2(r+\beta)} + \frac{5}{2}\frac{r\beta}{B^2(r+\beta)}(T_t - T_1) = \left(\frac{r}{r+\beta}\right)^{\frac{1}{2}}\psi_{\phi\phi\phi}, \tag{A 5}$$

$$\frac{\partial(\psi, A)}{\partial(\phi, r)} = \left(\frac{r}{r+\beta}\right)^{\frac{1}{2}} A_{\phi\phi}, \tag{A 6}$$

where ψ is a stream function, $u = \psi_\phi/r^2$, $v = -\psi_r/r$, and A in (A 6) represents either $T_t - T_1$ or $W = rw$. For the case $\beta \equiv 0$, these equations are exactly those solved in § 4 of I, and correspond to the jet structure when $\sigma = o(1)$, where there are no buoyancy effects.

Since both W and $T_t - T$ satisfy the same equation and they both vanish for $\phi \to \infty$, we write

$$T_t - T_1 = cW, \quad c = \text{constant}, \tag{A 7}$$

and look for solutions of the form

$$\psi = A(r)\,F(\xi), \quad W = B(r)\,G(\xi), \quad \xi = \phi/\delta(r). \tag{A 8}$$

Substitution of (A 8) into (A 5) and (A 6) yields two equations involving two functions, F and G, whose coefficients are functions of r through A and B. The term in (A 5) involving $T_t - T_1$ prevents a non-trivial solution for $A(r)$ and $B(r)$, and similarity of the type (A 8) is impossible unless $c \equiv 0$. In that case, the solution is

$$F(\xi) = \tanh\xi, \quad G(\xi) = \operatorname{sech}^2\xi$$

and

$$A(r) = 3H/4B(r) = \tfrac{1}{2}m\left[1 + \frac{36}{m^3(1+\beta)^{\frac{1}{2}}}\int_1^r \lambda^2 Q(\lambda)\,d\lambda\right]^{\frac{1}{3}},$$

$$\delta(r) = \frac{4A^2}{3Q}\left[\frac{r+\beta}{r(1+\beta)}\right]^{\frac{1}{2}},$$

$$Q(r) = \left[M^2 + H^2\frac{r-1}{r}\left(\frac{r+1+\beta}{r+\beta}\right)\right]^{\frac{1}{2}},$$

where m, M and H are respectively the mass, momentum and angular momentum flux of the jet at $r = 1$. Both m and H are easily found from the boundary-layer solution since, as § 6 notes, they are invariant through the turn at the intersection of $\mathscr{S}$ and $\mathscr{E}$. The value of M, however, depends on the details of this corner flow solution.

We note that, as $r \to \infty$,

$$A \sim (\tfrac{3}{2})^{\frac{1}{3}} \left(\frac{M^2+H^2}{1+\beta}\right)^{\frac{1}{3}} r,$$

$$\delta \sim (\tfrac{16}{3})^{\frac{1}{3}} \left(\frac{1+\beta}{M^2+H^2}\right)^{\frac{1}{3}}.$$

This similar solution for a neutrally buoyant jet obviously satisfies condition (6.3) with equality.

Appendix B

If $\sigma = o(1)$, the boundary layer on $\mathscr{S}$, as indicated in § 7, splits into a viscous and incompressible layer and also a thicker thermal layer; we write

$$r-1 = (\epsilon^{\frac{1}{2}}/\sigma)\, y_1$$

for the thermal-layer co-ordinate. If we also put $u = \epsilon^{\frac{1}{2}}\tilde{u}$ and $v = \sigma\chi\tilde{v}$ in (2.1), then the thermal-layer equations are

$$\frac{\partial(\tilde{\rho}\tilde{u})}{\partial y_1} + \chi \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\tilde{\rho}\tilde{v}\sin\theta) = 0, \tag{B 1}$$

$$\tilde{\rho}\tilde{u}\frac{\partial\tilde{T}}{\partial y_1} + \chi\tilde{\rho}\tilde{v}\frac{\partial\tilde{T}}{\partial\theta} = \frac{\partial}{\partial y_1}\left(\tilde{T}^{\omega}\frac{\partial\tilde{T}}{\partial y_1}\right), \tag{B 2}$$

$$\tilde{\rho}\tilde{u}\frac{\partial^2\tilde{v}}{\partial y_1^2} + \chi\tilde{\rho}\tilde{v}\frac{\partial^2\tilde{v}}{\partial\theta\,\partial y_1} = \frac{\alpha\epsilon^{\frac{1}{2}}}{\sigma^3\chi B^2}\frac{\partial\tilde{p}}{\partial\theta}, \tag{B 3}$$

$$\tilde{\rho}\tilde{T} = 1. \tag{B 4}$$

Now, the order of magnitude of χ is so far unspecified, but is clearly $\leqslant 1$. There are two cases to be considered as $\sigma \downarrow 0$ with $O(\epsilon)$ fixed.

Case (*a*). $O(\epsilon^{\frac{1}{6}}) < \sigma = o(1)$. Here, $\chi = \epsilon^{\frac{1}{2}}/\sigma^3$ from (B 3) so $\chi = o(1)$. The limiting form of (B 1) is

$$\partial(\tilde{\rho}\tilde{u})/\partial y_1 = 0$$

and (B 2) becomes

$$\tilde{\rho}\tilde{u}\frac{\partial\tilde{T}}{\partial y_1} = \frac{\partial}{\partial y_1}\left(\tilde{T}^{\omega}\frac{\partial\tilde{T}}{\partial y_1}\right).$$

Matching with the viscous layer gives $\tilde{\rho}\tilde{u} = -E_{\omega}(\theta)$, so with the match with the outer flow of § 2, we have

$$E_{\omega}\frac{\partial\tilde{T}}{\partial y_1} + \frac{\partial}{\partial y_1}\left(\tilde{T}^{\omega}\frac{\partial\tilde{T}}{\partial y_1}\right) = 0 \tag{B 5}$$

and $T = 1$ (or $T_{y_1} = 0$ for the insulated sphere) on $y_1 = 0$,

$$\tilde{T} \sim T_{\infty}(1+\beta) \quad \text{as} \quad y_1 \to \infty.$$

The solution is given in § 6 of I and in § 7 of this paper. Matching with the outer solution in this case is much as in § 4 and will not be repeated here; in any case, boundary conditions (4.3) and (4.4) on the outer flow are recovered.

Case (*b*). $O(\epsilon^{\frac{1}{4}}) < \sigma = o(\epsilon^{\frac{1}{5}})$. Regardless of the order of magnitude of χ, (B 3) shows T to be θ independent, so

$$\tilde{\rho}\tilde{u}\frac{d\tilde{T}}{dy_1} = \frac{d}{dy_1}\left(\tilde{T}^{\omega}\frac{d\tilde{T}}{dy_1}\right) \tag{B 6}$$

and (B 1) integrates to

$$\tilde{\rho}\tilde{v} = -\frac{1-\cos\theta}{\sin\theta}\frac{d\tilde{\rho}\tilde{u}}{dy_1} \quad \text{for} \quad 0 < \theta < \tfrac{1}{2}\pi. \tag{B 7}$$

The velocity components must match as $y_1 \to 0$ with the viscous layer, as $y_1 \to \infty$ with the outer flow, and as $\theta \to \frac{1}{2}\pi$ with the jet. Before matching as $\theta \to \frac{1}{2}\pi$, analysis of the region $\epsilon^{\frac{1}{2}} \times (\epsilon^{\frac{1}{2}}/\sigma)$ near the intersection of $\mathscr{S}$ and $\mathscr{E}$ must be done. Detailed analysis of that region shows that such a region is incapable of turning a θ-direction inflow 90° to be parallel with the fluid from the boundary-layer-eruption region (of size $\epsilon^{\frac{1}{2}} \times \epsilon^{\frac{1}{2}}$) beneath. Matching thus requires that $\tilde{\rho}\tilde{v} = 0$ on $\theta = \frac{1}{2}\pi$.

Therefore, from (B 7),

$$d(\tilde{\rho}\tilde{u})/dy_1 = 0, \tag{B 8}$$

the solution of which is $\tilde{\rho}\tilde{u}$ = constant. This solution *cannot* be matched pointwise with the viscous layer underneath. This matter is discussed in some detail in I, and the conclusion there is that

$$\tilde{\rho}\tilde{u} = -\int_0^{\frac{1}{2}\pi} E_{\omega}(\theta)\sin\theta\, d\theta. \tag{B 9}$$

The solution of (B 6) is then identical with solutions of (B 5) with $E_{\omega}(\theta)$ replaced by

$$\int_0^{\frac{1}{2}\pi} E_{\omega}(\theta)\sin\theta\, d\theta.$$

We notice then that, on matching with the outer expansion,

$$u_2 = -T_{\infty}(1+\beta)\int_0^{\frac{1}{2}\pi} E_{\omega}(\theta)\sin\theta\, \mathrm{d}\theta \quad \text{on} \quad \mathscr{S}. \tag{B 10}$$

The replacement of the pointwise matching to the viscous layer with the integrated form (B 9) is not approximate but exact; in I it is shown that there is an intermediate layer between the viscous and thermal layers that is inviscid and rotational, and which rearranges the mass inflow through the thermal layer to that required by the boundary-layer entrainment. (It turns out that this layer has a width of $(\epsilon/\sigma)^{\frac{2}{3}}$ as shown in § 7 of I.)

Noting the nature of the matching from (4.5), it is appropriate to choose $\chi\sigma = (\epsilon/\sigma)^{\frac{1}{3}}$, or $\chi = (\epsilon/\sigma^4)^{\frac{1}{3}} = o(1)$ for $\sigma > O(\epsilon^{\frac{1}{4}})$. Correcting $\tilde{T}$ and $\tilde{u}$ in (B 1)–(B 4) by terms of order χ, and denoting such terms by a prime, we have from (B 3)

$$\partial p'/\partial\theta = 0$$

and thus (B 2) becomes

$$(\tilde{\rho}u' + \rho'\tilde{u})\frac{dT'}{dy_1} = \frac{d^2}{dy_1^2}\tilde{T}^{\omega}T',$$

which shows that

$$\partial u'/\partial\theta = 0. \tag{B 11}$$

Careful matching of precisely the same form as in § 4 (with the outer expansion) gives

$$\lim_{y_1 \to \infty} [u'/y_1] = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_1 \sin\theta)|_{r=1}. \tag{B 12}$$

Equations (B 11) and (B 12) require

$$\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_1 \sin\theta)|_{r=1} = 0, \tag{B 13}$$

which cannot be true, in general, since the inviscid equations (2.7) are solved with $u_1 = 0$ on $r = 1$. The only solution that satisfies both (4.3) and (B 13) is $\mathbf{u}_1 \equiv 0$.

In summary, then, as $\sigma \downarrow 0$, the parameter range of case (*a*) is encountered first; though the boundary-layer structure is somewhat altered, the outer flow is unchanged from the $\sigma = O(1)$ case, and the matching is essentially the same. However, once σ is sufficiently small to lie in the range covered by case (*b*), the boundary layer and outer expansions will not match unless there is *no* free convection in the outer flow. The critical order of magnitude for σ is then $\epsilon^{\frac{1}{2}}$. If $\sigma/\epsilon^{\frac{1}{2}} \downarrow 0$ as $\epsilon \downarrow 0$, there is no free convection; if $\sigma/\epsilon^{\frac{1}{2}} \uparrow \infty$ as $\epsilon \downarrow 0$, the leading-order outer flow *is* free convection.

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